The distance domination of generalized de Bruijn and Kautz digraphs

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Abstract

Let $G = (V, A)$ be a digraph and $k \geq 1$ an integer. For $u, v \in V$, we say that the vertex u distance k-dominate v if the distance from u to v at most k. A set D of vertices in G is a distance k-dominating set if for each vertex of $V \setminus D$ is distance k-dominated by some vertex of D. The *distance k-domination number* of G, denoted by $\gamma_k(G)$, is the minimum cardinality of a distance k-dominating set of G. Generalized de Bruijn digraphs $G_B(n, d)$ and generalized Kautz digraphs $G_K(n, d)$ are good candidates for interconnection networks. Tian and Xu showed that $\lceil n/\sum_{j=0}^{k} d^j \rceil \leq \gamma_k(G_B(n,d)) \leq \lceil n/d^k \rceil$ and $\lceil n/\sum_{j=0}^{k} d^j \rceil \leq \gamma_k(G_K(n,d)) \leq \lceil n/d^k \rceil$. In this paper we prove that every generalized de Bruijn digraph $G_B(n, d)$ has the distance k-domination number $\lceil n/\sum_{j=0}^{k} d^j \rceil$ or $\lceil n/\sum_{j=0}^{k} d^j \rceil + 1$, and the distance k-domination number of every generalized Kautz digraph $G_K(n, d)$ bounded above by $\lceil n/(d^{k-1} + d^k) \rceil$. Additionally, we present various sufficient conditions for $\gamma_k(G_B(n,d)) = \lceil n/\sum_{j=0}^{k} d^j \rceil$ and $\gamma_k(G_K(n,d)) = \lceil n/\sum_{j=0}^{k} d^j \rceil$.

Keywords: Combinatorial problems; generalized de Bruijn digraph; generalized Kautz digraph; distance dominating set; dominating set

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1 Introduction

In this paper we deal with directed graphs (or digraphs) which admit self-loops but no multiple arcs. Unless otherwise defined, we follow [3, 10] for terminology and definitions. Let G be a digraph with *vertex set* $V(G)$ and *arc set* $A(G)$. If there is an arc from u to v, i.e., $(u,v) \in A(G)$, then v is called an *out-neighbor* of u; we also say that u *dominates* v. The *out-neighborhood* $O(u)$ of a vertex u is the set $\{v : (u,v) \in A(G)\}$. For $S \subseteq V(G)$, its *out-neighborhood* $O(S)$ is the set $\cup_{u\in S}O(u)$. Set $O_0(u) = \{u\}$ and $O_1(u) = O(u)$, we define recursively $O_i(u)$, called *i-th out-neighborhood* of u, by $\{O(O_{i-1}(u))\}$ for $i \geq 1$. The *i-th out-neighborhood* of S is the set $O_i(S) = \cup_{u\in S}O_i(u)$. The *closed out-neighborhood* $O[u]$ of u is the set $O(u) \cup \{u\}$, and $O[S]$ and $O_i[S]$ are defined analogously.

For $x, y \in V(G)$, the *distance* $d_G(x,y)$ from x to y is the length of an shortest (x,y)-directed path in G. Let k be a positive integer. A subset $D \subseteq V(G)$ is called a *distance k-dominating set* of G if for every vertex v of $V(G) \setminus D$, there is a vertex $u \in D$ such that $d_G(u,v) \leq k$, i.e., $\cup_{i=0}^{k}O_i(D) = V(G)$. The *distance k-domination number* of G, denoted by $\gamma_k(G)$, is the minimum cardinality of a distance k-dominating set of G. In particular, the distance 1-dominating set is the ordinary dominating set, which has been well studied [11].

Slater [11] termed a distance k-dominating set as a k-basis and also gave an interpretation for a k-basis in terms of communication networks. Since then many researchers pay much attention to this subject, for example [9, 19, 23]. The concept of distance domination in graphs finds applications in many structures and situations which give rise to graphs. A minimum distance k-dominating set of G may be used locate a minimum number of facilities (such as utilities, police stations, hospitals, transmission towers, blood banks, waste disposal dump) such that every intersection is within k city block of a facility. Barkauskas and Host [1] showed that the problem of determining $\gamma(G)$ is NP-hard for a general graph.

The network topology has a great impact on the system performance and reliability [26]. There are some well-known networks with good properties such as de Bruijn networks, Kautz networks and their generalizations (see, for example, [2, 4, 5, 13, 26]). Generalized de Bruijn and Kautz networks, denoted by $G_B(n,d)$ and $G_K(n,d)$ respectively, were introduced by Imase and Itoh [14]. The generalization removes the restriction on the cardinality of vertex set and make the network more general and valuable as a network model. A lot of features make it suitable for implementation of reliable networks. The most important feature such as small diameter [14], high connectivity [15], easy routing, and high reliability.

The generalized de Bruijn digraph $G_B(n,d)$ is defined by congruence equations as follows:

$$\begin{cases} V(G_B(n,d)) = \{0,1,2,\ldots,n-1\} \\ A(G_B(n,d)) = \{(x,y)\,|\,y \equiv dx+i \,(\mathrm{mod}\; n), 0 \le i \le d-1\}. \end{cases}$$

In particular, if $n=d^m$, then $G_B(n,d)$ is the de Bruijn digraph $B(d,m)$. The generalized Kautz digraph $G_K(n,d)$ is defined by following congruence equation:

$$\begin{cases} V(G_K(n,d)) = \{0,1,2,\ldots,n-1\} \\ A(G_K(n,d)) = \{(x,y)\,|\,y \equiv -dx-i \,(\mathrm{mod}\; n), 1 \le i \le d\}. \end{cases}$$

In particular, if $n=d^m+d^{m-1}$, then $G_K(n,d)$ is the Kautz digraph $K(d,m)$. The graphs $G_B(6,3)$ and $G_K(9,2)$ are exhibited in Fig. 1.

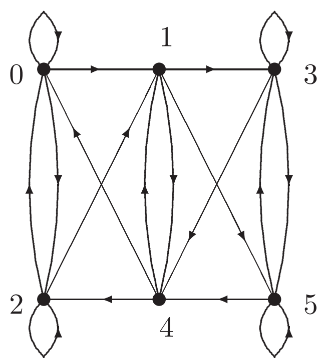

Figure 1 (a): $G_B(6,3)$

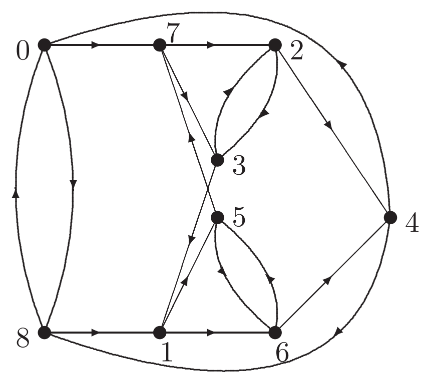

Figure 1 (b): $G_K(9,2)$

The structure properties of the generalized de Bruijn and Kautz digraphs receive more attention. Du et al. [6] studied the hamiltonian property of generalized de Bruijn and Kautz networks. Also, several structural objects such as spanning trees, Eulerian tours [17], closed walks [24] and small cycles [12] have been counted. Shan et al. [20, 21, 22] studied the absorbants and twin domination of generalized de Bruijn digraphs. Recently, Dong et al. [7] completely determined the domination number of generalized de Bruijn digraphs. Wang [27] showed that there is an efficient twin dominating set in $G_B(n,d)$ with $n=c(d+1)$ if and only if d is even and relatively prime to c. More studied progress on the generalized de Bruijn and Kautz networks can be found in [8, 25, 26].

In order to make our arguments easier to follow we introduce the *modulo interval* so as to represent the out-neighborhood of each vertex in $G_B(n,d)$ and $G_K(n,d)$. Let $I=\{0,1,\cdots,n-1\}$ denote the vertex set of $G_B(n,d)$. For any integers i,j satisfying $i \not\equiv j \pmod n$, a *modulo*

interval $[i, j]$ (mod n), with respect to modulo n, is defined by

$$[i,j] \,(\text{mod } n) = \begin{cases} \{i, i+1, \ldots, j\} \,(\text{mod } n) & \text{if } i \ (\text{mod } n) < j \ (\text{mod } n), \\ \{i, \ldots, n-1, 0, \ldots, j\} \,(\text{mod } n) & \text{if } i \ (\text{mod } n) > j \ (\text{mod } n). \end{cases}$$

By the definitions, $I = [0, n-1]$, and for each $j \in [0, n-1]$, clearly $O(j) = [jd, jd + (d-1)]$ (mod n) in $G_B(n, d)$ and $O(j) = [-jd - d, -jd - 1]$ (mod n) in $G_K(n, d)$.

Notice that if $d = 1$ then the graph $G_B(n, 1)$ (or $G_K(n, 1)$) has n self-loops. Throughout this paper, we always assume $d \geq 2$ and $n \geq d$. If the set $D = \{x, x+1, \cdots, x+k\}$ (mod n) is a dominating set or a distance k-dominating set of $G_B(n, d)$ (or $G_K(n, d)$), then D is called a *consecutive dominating set* or a *consecutive distance k-dominating set* of $G_B(n, d)$ (or $G_K(n, d)$). A *consecutive minimum dominating set* of $G_B(n, d)$ (or $G_K(n, d)$) is a consecutive dominating set with cardinality $\gamma(G_B(n, d))$ (or $\gamma(G_K(n, d))$) and a *consecutive distance k-dominating set* of $G_B(n, d)$ (or $G_K(n, d)$) is a consecutive distance k-dominating set with cardinality $\gamma_k(G_B(n, d))$ (or $\gamma_k(G_K(n, d))$).

Tian and Xu [25] established the upper and lower bounds on the distance k-domination number of $G_B(n, d)$ and $G_K(n, d)$. This paper continues to study distance k-domination in generalized de Bruijn and Kautz digraphs. In Subsection 2.1, we show that every generalized de Bruijn digraph $G_B(n, d)$ has the distance k-domination number either $\lceil n / \sum_{j=0}^{k} d^j \rceil$ or $\lceil n / \sum_{j=0}^{k} d^j \rceil + 1$. In Subsection 2.2, we derive various sufficient conditions for $\gamma_k(G_B(n, d)) = \lceil n / \sum_{j=0}^{k} d^j \rceil$. In Section 3, we gives a sharp upper bound of $\gamma_k(G_K(n, d))$, which improves the previous upper bound of $\gamma_k(G_K(n, d))$, due to Tian and Xu [25]. In closing section, we pose two open problems.

2 The minimum distance k-dominating sets in $G_B(n, d)$

In the first subsection of this section, by constructing a distance k-dominating set of an arbitrary generalized de Bruijn digraph $G_B(n, d)$, we show that the distance k-domination number of $G_B(n, d)$ has exactly two values. In next subsection, we describe various sufficient conditions for the distance k-domination number equal to one of two values.

2.1 The distance k-domination number of $G_B(n, d)$

Tian and Xu [25] observed the following upper and lower bounds on $\gamma_k(G_B(n, d))$.

Lemma 2.1. ([25]) *For every generalized de Bruijn digraph $G_B(n,d)$,*

$$\left\lceil n\Big/\sum_{j=0}^{k} d^j \right\rceil \le \gamma_k(G_B(n,d)) \le \left\lceil \frac{n}{d^k} \right\rceil.$$

We are ready to improve the above upper bound on $\gamma_k(G_B(n,d))$ by directly constructing a (consecutive) distance k-dominating set of $G_B(n,d)$ with cardinality $\lceil n/(\sum_{j=0}^{k} d^j)\rceil + 1$. The following lemma plays a key role in constructing such a distance k-dominating set of $G_B(n,d)$.

Lemma 2.2. *Every generalized de Bruijn digraph $G_B(n,d)$ contains a vertex x satisfying the following inequality:*

$$x + \left\lceil n\Big/\sum_{j=0}^{k} d^j \right\rceil - (d-2) \le dx \le x + \left\lceil n\Big/\sum_{j=0}^{k} d^j \right\rceil \pmod n. \tag{1}$$

Proof. We choose an arbitrary vertex $x \in V(G_B(n,d))$. If x satisfies (1), we are done. Otherwise, the vertex x clearly satisfies either

$$0 \le dx \le x + \left\lceil n/\sum_{j=0}^{k} d^j \right\rceil - (d-1) \pmod n$$

or

$$x + \left\lceil n/\sum_{j=0}^{k} d^j \right\rceil + 1 \le dx \le n-1 \pmod n.$$

We find the desired vertex by distinguishing the following two cases.

Case 1. $0 \le dx \le x + \lceil n/\sum_{j=0}^{k} d^j \rceil - (d-1) \pmod n$. Note that if x increases by integer i, then the value of dx is increased to $d(x+i) = dx + di$. In this case, we find the desired vertex by increasing the value of x. Since $dx \le x + \lceil n/\sum_{j=0}^{k} d^j \rceil - (d-1) \pmod n$, there exists an integer i (≥ 0) such that x and i satisfy the following inequality

$$d(x+i) \le x + \left\lceil n\Big/\sum_{j=0}^{k} d^j \right\rceil - (d-2) \pmod n, \tag{2}$$

since $i = 0$ satisfies the inequality. Let i be the maximal integer satisfying (2). We claim that

$$d(x+i) \ge (x+i) + \left\lceil n\Big/\sum_{j=0}^{k} d^j \right\rceil - 2(d-2) \pmod n. \tag{3}$$

Indeed, if $d(x+i) \le (x+i) + \left\lceil n \big/ \sum_{j=0}^{k} d^j \right\rceil - 2(d-2) - 1 \pmod n$, then

$$d(x+i+1) \le (x+i+1) + \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil - (d-2) \pmod n.$$

So $i+1$ satisfies (2) too, this contradicts the maximality of i. Hence (3) follows. If the equality holds in (2), that is,

$$d(x+i) = x + \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil - (d-2) \pmod n,$$

then $x+i$ satisfies (1). So we replace x by $x+i$, and obtain the desired vertex. Otherwise, by (3), we have

$$(x+i) + \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil - 2(d-2) \le d(x+i) \le (x+i) + \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil - (d-1) \pmod n.$$

Hence,

$$(x+i+1) + \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil - (d-3) \le d(x+i+1) \le (x+i+1) + \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil \pmod n.$$

Clearly, $x+i+1$ satisfies (1). Thus we replace x by $x+i+1$ and obtain the desired vertex.

Case 2. $x + \left\lceil n \big/ \sum_{j=0}^{k} d^j \right\rceil + 1 \le dx \le n-1 \pmod n$. We can obtain the desired vertex by decreasing the value of x. Clearly, there exists an integer i (≥ 0) such that x and i satisfy the following inequality

$$d(x-i) \ge (x-i) + \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil \pmod n, \tag{4}$$

since the inequality $dx \ge x + \left\lceil n \big/ \sum_{j=0}^{k} d^j \right\rceil + 1$ implies that $i = 0$ satisfies (4). Let i be the maximal integer satisfying (4). We claim that

$$d(x-i) \le (x-i) + \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil + d - 2 \pmod n. \tag{5}$$

Suppose, to the contrary, that $d(x-i) \geq (x-i) + \lceil n/\sum_{j=0}^{k} d^j \rceil + d - 1 \pmod{n}$. Equivalently,

$$d(x-(i+1)) \geq (x-(i+1)) + \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil \pmod{n}.$$

But then $i+1$ satisfies (4). This contradicts the maximality of i. Thus (5) holds. If the equality holds in (4), then the vertex $x-i$ satisfies (1). So we obtain the desired vertex by replacing x by $x-i$. Otherwise, by (5), we have

$$(x-i) + \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil + 1 \leq d(x-i) \leq (x-i) + \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil + d - 2 \pmod{n}.$$

Hence,

$$\begin{aligned}(x-(i+1)) + \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil - (d-2) &\leq d(x-(i+1)) \\ &\leq (x-(i+1)) + \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil - 1 \pmod{n}.\end{aligned}$$

Hence $x-(i+1)$ satisfies (1). We obtain the desired vertex by replacing x by $x-(i+1)$. $\square$

Theorem 2.1. *For every generalized de Bruijn digraph $G_B(n,d)$,*

$$\gamma_k(G_B(n,d)) = \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil \ \text{or}\ \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil + 1.$$

Proof. By Lemma 2.1, it suffices to show that $\gamma(G_B(n,d)) \leq \lceil n/\sum_{j=0}^{k} d^j \rceil + 1$. The proof is by directly constructing a (consecutive) distance k-dominating set of $G_B(n,d)$ with cardinality $\lceil n/(\sum_{j=0}^{k} d^j) \rceil + 1$. By Lemma 2.2, there is a vertex x in $G_B(n,d)$ that satisfies (1). Let $D = \{x, x+1, \ldots, x + \lceil n/\sum_{j=0}^{k} d^j \rceil\}$. We show that D is a distance k-dominating set of $G_B(n,d)$. By the definition, we need to prove that $\bigcup_{i=0}^{k} O_i(D) = V(G_B(n,d))$.

First, we show that the vertices of $O_{i-1} \cup O_i(D)$ are consecutive for all i, $1 \leq i \leq k$. The out-neighborhoods of vertices in D are given as follows.

$$O(x) = \{dx, dx+1, \ldots, dx+d-1\} \pmod{n},$$

$$O(x+1) = \{d(x+1), d(x+1)+1, \ldots, d(x+1)+d-1\} \pmod{n},$$

$$\vdots$$

$$O\Big(x + \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil\Big) = \Big\{d\Big(x + \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil\Big), \ldots, d\Big(x + \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil\Big) + d - 1\Big\} \pmod{n}.$$

Then $O(D) = \left[dx, d\left(x + \lceil n/\sum_{j=0}^{k} d^j \rceil\right) + d - 1\right] \pmod n$. Similarly, the i-th out-neighborhoods $O_i(D) = \left[d^i x, d^i\left(x + \lceil n/\sum_{j=0}^{k} d^j \rceil\right) + (d-1)\sum_{j=0}^{i} d^j\right] \pmod n$ for each $i, 1 \le i \le k$. Since x satisfying the inequality (1), there exists an integer h, $0 \le h \le d-2$, such that $dx = x + \lceil n/\sum_{j=0}^{k} d^j \rceil - h \pmod n$, so we have

$$
\begin{aligned}
d^2 x &= d\Big(x + \Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil\Big) - dh \pmod n,\\
d^3 x &= d^2\Big(x + \Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil\Big) - d^2 h \pmod n,\\
&\ \vdots\\
d^k x &= d^{k-1}\Big(x + \Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil\Big) - d^{k-1} h \pmod n.
\end{aligned}
$$

Thus $O_{i-1}(D) \cap O_i(D) \ne \emptyset$ for all $i, 1 \le i \le k$. This implies that the vertices of $O_{i-1}(D) \cup O_i(D)$ are consecutive, since the vertices of $O_i(D)$ are consecutive for each i, $0 \le i \le k$. Therefore, the vertices of $\bigcup_{i=0}^{k} O_i(D)$ are consecutive.

Next we show that $\bigcup_{i=0}^{k} O_i(D)$ contains all the vertices of $G_B(n, d)$. Note that $O_1(D) \cap D \ne \emptyset$. Thus it suffices to show that $O_k(D) \cap D \ne \emptyset$. For the last vertex in $O_k(D)$, since x satisfies (1), we have

$$
\begin{aligned}
&d^k\Big(x + \Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil\Big) + (d-1)\sum_{j=0}^{k} d^j\\
&= d^{k-1}\Big(x + \Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil - h\Big) + d^k \Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil + \big(d-1\big)\sum_{j=0}^{k} d^j\\
&= d^{k-1}x + \big(d^k + d^{k-1}\big)\Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil + (d-1)d^k - hd^{k-1} + \big(d-1\big)\sum_{j=0}^{k} d^j\\
&\quad \vdots\\
&= x + \Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil \sum_{j=0}^{k} d^j - h\sum_{j=0}^{k-1} d^j + \big(d-1\big)\sum_{j=0}^{k} d^j\\
&= x + (d-1) + \Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil \sum_{j=0}^{k} d^j + \big(d(d-1) - h\big)\sum_{j=0}^{k-1} d^j\\
&\ge x \pmod n
\end{aligned}
$$

The last inequality holds, since $d \geq 2$ and $0 \leq h \leq d-2$. Hence $O_k(D) \cap D \neq \emptyset$, and so

$$\bigcup_{i=1}^{k} O_i(D) \supseteq \{x + \lceil n/\sum_{j=0}^{k} d^j \rceil, \ldots, n-1, 0, 1, \ldots, x\}.$$

This implies that $\bigcup_{i=0}^{k} O_i(D) = V(G_B(n,d))$, that is, D is a (consecutive) distance k-dominating set of $G_B(n,d)$. Consequently, $\gamma_k(G_B(n,d)) \leq |D| = \lceil n/\sum_{j=0}^{k} d^j \rceil + 1$. □

For distance $k=1$ we obtain the following result.

Corollary 2.1. ([7]) *For every generalized de Bruijn digraph $G_B(n,d)$, either $\gamma(G_B(n,d)) = \lceil \frac{n}{d+1} \rceil$ or $\gamma(G_B(n,d)) = \lceil \frac{n}{d+1} \rceil + 1$.*

2.2 The generalized de Bruijn digraphs $G_B(n,d)$ with $\gamma(G_B(n,d)) = \lceil \frac{n}{d+1} \rceil$

In the next subsection, we derive various sufficient conditions for the distance k-domination number to achieve the value $\lceil n/\sum_{j=0}^{k} d^j \rceil$ in a generalized de Bruijn digraph $G_B(n,d)$.

Theorem 2.2. *If there exists a vertex $x \in V(G_B(n,d))$ satisfying the following congruence equation:*

$$(d-1)x \equiv \left\lceil n \Big/ \sum_{j=0}^{k} d^j \right\rceil - h \pmod{n}, \tag{6}$$

for some h where $0 \leq \left(\sum_{j=0}^{k-1} d^j\right)h \leq \left(\sum_{j=0}^{k} d^j\right)\lceil n/\sum_{j=0}^{k} d^j \rceil - n$, then $\gamma_k(G_B(n,d)) = \lceil n/\sum_{j=0}^{k} d^j \rceil$, and $D = \left\{x, x+1, x+2, \ldots, x + \lceil n/\sum_{j=0}^{k} d^j \rceil - 1\right\}$ is a consecutive minimum distance k-dominating set of $G_B(n,d)$.

Proof. Let x be a vertex of $G_B(n,d)$ satisfying Eq. (6). Note that $|D| = \lceil n/\sum_{j=0}^{k} d^j \rceil$. By Theorem 2.1, it is sufficient to show that $D = \{x, x+1, x+2, \ldots, x + \lceil n/\sum_{j=0}^{k} d^j \rceil - 1\}$ is a distance k-dominating set of $G_B(n,d)$. For this purpose, we show that $\bigcup_{i=1}^{k} O_i(D) = V(G_B(n,d))$.

We first prove that the vertices of $O_{i-1}(D) \cup O_i(D)$ are consecutive for all $i, 1 \leq i \leq k$. By

the definition of $G_B(n,d)$, the out-neighborhoods $O(D)$ of D are given as follows.

$$
\begin{aligned}
&O(x) = \{dx, dx+1, \ldots, dx+d-1\} \pmod n,\\
&O(x+1) = \{d(x+1), d(x+1)+1, \ldots, d(x+1)+d-1\} \pmod n,\\
&\quad\vdots\\
&O\Big(x + \Big\lceil n\Big/\sum_{j=0}^{k} d^j\Big\rceil - 1\Big) = \Big\{d\Big(x+\Big\lceil n\Big/\sum_{j=0}^{k} d^j\Big\rceil\Big) - d, d\Big(x+\Big\lceil n\Big/\sum_{j=0}^{k} d^j\Big\rceil\Big)\\
&\qquad\qquad - d + 1, \ldots, d\Big(x+\Big\lceil n\Big/\sum_{j=0}^{k} d^j\Big\rceil\Big) - 1\Big\} \pmod n.
\end{aligned}
$$

Then $O(D) = \big[dx, dx + d\lceil n/\sum_{j=0}^{k} d^j\rceil - 1\big] \pmod n$. Similarly, we have $O_i(D) = [d^i x, d^i\big(x + \lceil n/\sum_{j=0}^{k} d^j\rceil\big) - 1] \pmod n$. Clearly, $|O_i(D)| = d^i\lceil n/\sum_{j=0}^{k} d^j\rceil$ for all $i, 0 \le i \le k$. Since x satisfies Eq. (6), we have

$$
\begin{aligned}
O(D) &= \Big[x + \Big\lceil n/\sum_{j=0}^{k} d^j\Big\rceil - h, d\Big(x+\Big\lceil n\Big/\sum_{j=0}^{k} d^j\Big\rceil\Big) - 1\Big] \pmod n,\\
O_2(D) &= \Big[d\Big(x+\Big\lceil n\Big/\sum_{j=0}^{k} d^j\Big\rceil\Big) - dh, d^2\Big(x+\Big\lceil n\Big/\sum_{j=0}^{k} d^j\Big\rceil\Big) - 1\Big] \pmod n,\\
&\quad\vdots\\
O_k(D) &= \Big[d^{k-1}\Big(x+\Big\lceil n\Big/\sum_{j=0}^{k} d^j\Big\rceil\Big) - d^{k-1}h, d^k\Big(x+\Big\lceil n\Big/\sum_{j=0}^{k} d^j\Big\rceil\Big) - 1\Big] \pmod n.
\end{aligned}
$$

Hence it can be seen that $|O_{i-1}(D) \cap O_i(D)| = d^{i-1}h$ for all $i, 1 \le i \le k$. Note that the vertices of each $O_i(D)$ $(i \ge 0)$ are consecutive. By the above observations, if $h = 0$, then the last vertex of $O_{i-1}(D)$ and the first vertex of $O_i(D)$ are consecutive; while if $h > 0$, then $O_{i-1}(D) \cap O_i(D) \ne \emptyset$. Thus the vertices of $O_{i-1}(D) \cup O_i(D)$ are consecutive for all $i, 1 \le i \le k$.

We next show that $\bigcup_{i=0}^{k} O_i(D) = V(G_B(n,d))$. As observed above, we see that the vertices of $\bigcup_{i=0}^{k} O_i(D)$ are consecutive, In particular, the vertices of $D \cup O_1(D)$ are consecutive. Thus it suffices to show that the vertices $O_k(D) \cup D$ are consecutive. For the last vertex in $O_k(D)$,

because $0 \le \big(\sum_{j=0}^{k-1} d^j\big)h \le \big(\sum_{j=0}^{k} d^j\big)\lceil n/\sum_{j=0}^{k} d^j\rceil - n$, we have

$$\begin{aligned}
& d^k\Big(x + \Big\lceil n\Big/\sum_{j=0}^{k} d^j\Big\rceil\Big) - 1 \pmod n \\
&= x + \Big(\sum_{j=0}^{k} d^j\Big)\Big\lceil n\Big/\sum_{j=0}^{k} d^j\Big\rceil - \Big(\sum_{j=0}^{k-1} d^j\Big)h - 1 \pmod n \text{ (by (6))} \\
&\ge x - 1 \pmod n.
\end{aligned}$$

This implies that the vertices of $O_k(D) \cup D$ are consecutive, so

$$\bigcup_{i=1}^{k} O_i(D) \supseteq \big\{x + \lceil n/\sum_{j=0}^{k} d^j\rceil, \ldots, n-1, 0, 1, \ldots, x-1\big\}.$$

This implies that $\bigcup_{i=0}^{k} O_i(D) = V(G_B(n,d))$, hence D is a distance k-dominating set of $G_B(n,d)$. This complete the proof of Theorem 2.2. $\square$

As a special case of Theorem 2.2, we immediately have the following corollary.

Corollary 2.2. *Let $\sum_{j=0}^{k} d^j \mid n$. If there is a vertex $x \in V(G_B(n,d))$ satisfying congruence equation:*

$$(d-1)x \equiv n\Big/\sum_{j=0}^{k} d^j \pmod n, \tag{7}$$

then $\gamma_k(G_B(n,d)) = n/\sum_{j=0}^{k} d^j$ and $D = \{x, x+1, \cdots, x + n/\sum_{j=0}^{k} d^j - 1\}$ is a consecutive minimum distance k-dominating set of $G_B(n,d)$.

Remark 2.1. If $G_B(n,d)$ contains no vertex x satisfying (6) in Theorem 2.2, it is possible to encounter $\gamma_k(G_B(n,d)) = \lceil n/\sum_{j=0}^{k} d^j\rceil + 1$. For example, let $G_B(40,3)$ and $k = 3$. The congruence equation $(d-1)x \equiv \lceil n/\sum_{j=0}^{k} d^j\rceil - h \pmod n$ is $2x \equiv 1 \pmod{40}$ where $h = 0$, since $40/\sum_{j=0}^{3} 3^j = 1$. Clearly, there is no vertex satisfying $2x \equiv 1 \pmod{40}$. We can deduce that $\gamma_3(G_B(40,3)) = \lceil 40/\sum_{j=0}^{3} 3^j\rceil + 1 = 2$. Indeed, for each x of $G_B(40,3)$, it can be verify that $\{x\}$ is not a distance 3-dominating set of $G_B(40,3)$ by simply enumeration.

Recalling that $G_B(d^m, d) = B(d,m)$ when $n = d^m$. For cases $k = 1$ and $k = 2$, the distance k-domination numbers of a de Bruijn digraph $B(d,m)$ were proved by Araki [1] and Tian [25], respectively. As an application of Theorem 2.2, we can determine the distance k-domination number of a de Bruijn digraph for all $k \ge 1$.

Corollary 2.3. *For $d \geq 2$, $\gamma_k(B(d,m)) = \left\lceil d^m / \sum_{j=0}^{k} d^j \right\rceil$.*

Proof. If $m \leq k$, then clearly $\gamma_k(B(d,m)) = \gamma_k(G_B(d^m, d)) = 1 = \lceil d^m / \sum_{j=0}^{k} d^j \rceil$ by Theorem 2.2, so the assertion holds. We may therefore assume $m > k$. Let $m = ik + l$, where $i \geq 1$ and $0 \leq l \leq k-1$. Note that $d^m = (\sum_{j=0}^{k} d^j)(d^{m-k} - d^{m-k-1}) + d^{m-k-1}$, $d^{m-k-1} = (\sum_{j=0}^{k} d^j)(d^{m-2k-1} - d^{m-2k-2}) + d^{m-2k-2}$, $\cdots$, then we have

$$d^m = \begin{cases} (\sum_{j=0}^{k} d^j)[(d^{m-k} - d^{m-k-1}) + (d^{m-2k-1} - d^{m-2k-2}) \\ \qquad + \cdots + (d^{m-(i-1)k-(i-2)} - d^{m-(i-1)k-(i-1)})] + d^{m-(i-1)k-(i-1)}, \text{ if } l < i, \\ (\sum_{j=0}^{k} d^j)[(d^{m-k} - d^{m-k-1}) + (d^{m-2k-1} - d^{m-2k-2}) \\ \qquad + \cdots + (d^{m-ik-(i-1)} - d^{m-ik-i})] + d^{m-ik-i}, \quad \text{if } l \geq i. \end{cases}$$

Because $m = ik + l$ and $0 \leq l \leq k-1$, if $l < i$, then $d^{m-(i-1)k-(i-1)} = d^{l+k-(i-1)} \leq d^k$; and if $l \geq i$, then $d^{m-ik-i} = d^{l-i} < d^k$. Thus

$$\left\lceil d^m \Big/ \sum_{j=0}^{k} d^j \right\rceil = \begin{cases} (d-1)(d^{m-k-1} + d^{m-2k-2} + \cdots + d^{m-(i-1)k-(i-1)}) + 1, \text{ if } l < i, \\ (d-1)(d^{m-k-1} + d^{m-2k-2} + \cdots + d^{m-ik-i}) + 1, \text{ if } l \geq i. \end{cases}$$

Hence either $x = d^{m-k-1} + d^{m-2k-2} + \cdots + d^{m-(i-1)k-(i-1)}$ or $x = d^{m-k-1} + d^{m-2k-2} + \cdots + d^{m-ik-i}$ in $B(d,m)$ satisfies the congruence equation $(d-1)x \equiv \lceil d^m / \sum_{j=0}^{k} d^j \rceil - h \pmod{n}$ where $h = 1$ and $0 \leq h \sum_{j=0}^{k-1} d^j \leq (\sum_{j=0}^{k} d^j) \lceil d^m / \sum_{j=0}^{k} d^j \rceil - d^m$. Therefore, $\gamma_k(B(d,m)) = \lceil d^m / \sum_{j=0}^{k} d^j \rceil$ by Theorem 2.2. $\square$

As an application of Corollary 2.2, we provide a new sufficient condition for $\gamma_k(G_B(n,d))$ equal to $\lceil n / \sum_{j=0}^{k} d^j \rceil$. For this purpose, we need the following result in elementary number theory.

For notational convenience, $m \,|\, n$ means that m divides n and $m \nmid n$ means that m does not divide n where m, n are integers. For integers $a_1, a_2, \ldots, a_n$, the *greatest common divisor* of $a_1, a_2, \ldots, a_n$ is denoted by $(a_1, a_2, \ldots, a_n)$.

Lemma 2.3. ([18]) *For integers $a_1, a_2, \ldots, a_m$ $(m \geq 1)$, b and n, the congruence equation $\sum_{i=1}^{m} a_i x_i \equiv b \pmod{n}$ has at least a solution if and only if $(a_1, a_2, \ldots, a_m, n) \,|\, b$.*

Theorem 2.3. *For every generalized de Bruijn digraph $G_B(n,d)$, if both n and d satisfy one of the following conditions:*

(i) $\sum_{j=0}^{k} d^j \,|\, n$ *and* $(d-1, n) \,|\, n / \sum_{j=0}^{k} d^j$,

(ii) $\lceil n/\sum_{j=0}^{k} d^j \rceil \equiv q \pmod{(d-1,n)}$, *where* q *satisfies the inequality* $0 \le q(\sum_{j=0}^{k-1} d^j) \le (\sum_{j=0}^{k} d^j)\lceil n/\sum_{j=0}^{k} d^j \rceil - n$,

then $\gamma_k(G_B(n,d)) = \lceil n/\sum_{j=0}^{k} d^j \rceil$ *and there is a vertex* $x \in V(G_B(n,d))$ *such that* $D = \{x, x+1, \cdots, x+\lceil n/\sum_{j=0}^{k} d^j \rceil - 1\}$ *is a consecutive minimum distance* k*-dominating set of* $G_B(n,d)$.

Proof. Let n and d satisfy one of the conditions (i)-(ii). We show that $G_B(n,d)$ contains a vertex x such that $D = \{x, x+1, \cdots, x+\lceil n/\sum_{j=0}^{k} d^j \rceil - 1\}$ is a consecutive minimum distance k-dominating set of $G_B(n,d)$. By Theorem 2.2, it suffices to show that there exists a vertex $x \in V(G_B(n,d))$ satisfies $(d-1)x \equiv \left\lceil n\Big/\sum_{j=0}^{k} d^j \right\rceil - h \pmod n$ (Eq. (6)) for some h where $0 \le \big(\sum_{j=0}^{k-1} d^j\big)h \le \big(\sum_{j=0}^{k} d^j\big)\lceil n/\sum_{j=0}^{k} d^j \rceil - n$.

(i) Suppose that $\sum_{j=0}^{k} d^j \mid n$ and $(d-1,n) \mid n/\sum_{j=0}^{k} d^j$. By Lemma 2.3, there is a vertex $x \in V(G_B(n,d))$ satisfying $(d-1)x \equiv n/\sum_{j=0}^{k} d^j \pmod n$, so the assertion follows directly from Corollary 2.2.

(ii) Suppose that $\lceil n/\sum_{j=0}^{k} d^j \rceil \equiv q \pmod{(d-1,n)}$, where q satisfies the inequality $0 \le q(\sum_{j=0}^{k-1} d^j) \le (\sum_{j=0}^{k} d^j)\lceil n/\sum_{j=0}^{k} d^j \rceil - n$. Let $(d-1,n) = r$ and $\lceil n/\sum_{j=0}^{k} d^j \rceil = pr + q$ where $p \ge 0$ and $0 \le q \le r-1$. Set $q = h$. Since $(d-1,n)|pr$, the equation $(d-1)x \equiv pr \pmod n$ has a solution by Lemma 2.3. Hence, there exists a vertex $x \in V(G_B(n,d))$ satisfying $(d-1)x \equiv \lceil n/\sum_{j=0}^{k} d^j \rceil - h \pmod n$, as desired. $\Box$

By applying Theorems 2.1 and 2.2, we obtain the following sufficient condition for $\gamma_k(G_B(n,d))$ equal to $\lceil n/\sum_{j=0}^{k} d^j \rceil$.

Theorem 2.4. *If* $n = p(\sum_{j=0}^{k} d^j) + q$, *where* $p \ge 1$ *and* $1 \le q \le \min\{1 + 2\sum_{j=0}^{k-1} d^j, \sum_{j=1}^{k} d^j\}$, *then* $\gamma_k(G_B(n,d)) = \lceil n/\sum_{j=0}^{k} d^j \rceil$.

Proof. By Theorem 2.1, we have known that $G_B(n,d)$ contains a vertex satisfying (1). Let x be such a vertex and let $D = \{x, x+1, \cdots, x+\lceil n/\sum_{j=0}^{k} d^j \rceil - 1\}$. We claim that D is a distance k-dominating set of $G_B(n,d)$. By the definition, it suffices to show that $\bigcup_{i=0}^{k} O_i(D) = V(G_B(n,d))$.

As before, we first show the vertices of $O_{i-1}(D) \cup O_i(D)$ are consecutive for all $i, 1 \le i \le k$. As already observed in Theorem 2.2, we have $O_i(D) = [d^i x, d^i\big(x + \lceil n/\sum_{j=0}^{k} d^j \rceil\big) - 1] \pmod n$ and $|O_i(D)| = d^i \lceil n/\sum_{j=0}^{k} d^j \rceil$ for all $i, 0 \le i \le k$. Since x satisfies the inequality (1), there

exists an integer h, $0 \le h \le d-2$ such that $dx = x + \lceil n/\sum_{j=0}^{k} d^j \rceil - h \pmod n$.

$$
\begin{aligned}
d^2x &= d\Big(x + \Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil\Big) - dh \pmod n,\\
d^3x &= d^2\Big(x + \Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil\Big) - d^2h \pmod n,\\
&\ \ \vdots\\
d^kx &= d^{k-1}\Big(x + \Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil\Big) - d^{k-1}h \pmod n.
\end{aligned}
$$

Since $O_i(D) = [d^ix, d^i\big(x + \lceil n/\sum_{j=0}^{k} d^j \rceil\big) - 1] \pmod n$ for all $i, 0 \le i \le k$, the vertices of $O_{i-1}(D) \cap O_i(D) \neq \emptyset$ are consecutive for all $i, 1 \le i \le k$.

By the above fact, we show that $\bigcup_{i=1}^{k} O_i(D)$ contains all the vertices of $G_B(n,d) \setminus D$ by showing the vertices of $O_k(D) \cup D$ are consecutive. We consider the last vertex in $O_k(D)$. Since $n = p(\sum_{j=0}^{k} d^j) + q$, $\lceil n/\sum_{j=0}^{k} d^j \rceil \sum_{j=0}^{k} d^j = n - q + \sum_{j=0}^{k} d^j$. Hence, by $dx = x + \lceil n/\sum_{j=0}^{k} d^j \rceil - h \pmod n$ where $0 \le h \le d-2$, we have

$$
\begin{aligned}
d^kx + d^k\Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil - 1 &= d^{k-1}\Big(x + \Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil - h\Big) + d^k\Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil - 1\\
&= d^{k-1}x + (d^k + d^{k-1})\Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil - d^{k-1}h - 1\\
&= \cdots\\
&= (x-1) + \Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil \sum_{j=0}^{k} d^j - h\sum_{j=0}^{k-1} d^j \pmod n\\
&= (x-1) + 1 + (d-h)\sum_{j=0}^{k-1} d^j - q \pmod n\\
&\ge (x-1) + 1 + 2\sum_{j=0}^{k-1} d^j - q \pmod n\\
&\ge x - 1,
\end{aligned}
$$

The last inequality holds, since $1 \le q \le \min\big\{1 + 2\sum_{j=0}^{k-1} d^j, \sum_{j=1}^{k} d^j\big\}$. Note that the vertices of $O_i(D)$ are consecutive for all $i, 0 \le i \le k$, so $\bigcup_{i=1}^{k} O_i(D) \supseteq \{x + \lceil n/\sum_{j=0}^{k} d^j \rceil, \ldots, n-1, 0, 1, \ldots, x-1\}$. This implies that $\bigcup_{i=1}^{k} O_i(D) \supseteq V(G_B(n,d)) \setminus D$, hence $D = \{x, x+1, x+$

$2,\ldots,x+\lceil n/\sum_{j=0}^{k}d^j\rceil-1\}$ is a distance k-dominating set of $G_B(n,d)$. Thus $\gamma_k(G_B(n,d))\leq |D|=\lceil n/\sum_{j=0}^{k}d^j\rceil$. By Theorem 2.1, $\gamma_k(G_B(n,d))=\lceil n/\sum_{j=0}^{k}d^j\rceil$. $\square$

3 The minimum distance k-dominating sets in $G_K(n,d)$

Tian and Xu [25] observed the following upper and lower bounds on $\gamma_k(G_K(n,d))$.

Lemma 3.1. ([25]) *For any generalized Kautz digraph $G_K(n,d)$,*

$$\left\lceil n\Big/\sum_{j=0}^{k}d^j\right\rceil \leq \gamma_k(G_K(n,d)) \leq \left\lceil \frac{n}{d^k}\right\rceil.$$

In this section, we shall improve the above upper bound on $\gamma_k(G_K(n,d))$ by constructing a consecutive distance k-dominating set of $G_K(n,d)$.

Theorem 3.1. *Let $G_K(n,d)$ be a generalized Kautz digraph. Then $D=\big\{0,1,\cdots,\lceil n/(d^k+d^{k-1})\rceil-1\big\}$ is a distance k-dominating set of $G_K(n,d)$, and so*

$$\gamma_k(G_K(n,d))\leq\left\lceil\frac{n}{d^k+d^{k-1}}\right\rceil.$$

Proof. We show that D is a distance k-dominating set of $G_K(n,d)$. By the definitions of $G_K(n,d)$ and i-th out-neighborhood, if k is odd, then we obtain

$$O_{k-1}(D)=\big\{0,1,\cdots,d^{k-1}\lceil n/(d^k+d^{k-1})\rceil-1\big\},$$
$$O_k(D)=\big\{n-1,n-2,\cdots,n-d^k\lceil n/(d^k+d^{k-1})\rceil\big\};$$

if k is even, then

$$O_{k-1}(D)=\big\{n-1,n-2,\cdots,n-d^{k-1}\lceil n/(d^k+d^{k-1})\rceil\big\},$$
$$O_k(D)=\big\{0,1,\cdots,d^k\lceil n/(d^k+d^{k-1})\rceil-1\big\}.$$

In both cases, we have $|O_{k-1}(D)|=d^{k-1}\lceil n/(d^k+d^{k-1})\rceil$ and $|O_k(D)|=d^k\lceil n/(d^k+d^{k-1})\rceil$. Note that the vertices of $O_{k-1}(D)$ and $O_k(D)$ are consecutive, and $(d^k+d^{k-1})\lceil n/(d^k+d^{k-1})\rceil\geq n$, so $O_{k-1}(D)\cup O_k(D)=V(G_K(n,d))$. Hence D is a distance k-dominating set of $G_K(n,d)$. Therefore, $\gamma_k(G_K(n,d))\leq|D|=\lceil n/(d^k+d^{k-1})\rceil$. $\square$

Remark 3.1. The upper bound on the distance k-domination number given in Theorem 3.1 is sharp. For example, we consider the digraph $G_K(7,2)$. We claim that $\gamma_2(G_K(7,2)) = 2 = \lceil \frac{7}{2+4} \rceil$. Suppose not, we have $\gamma_2(G_K(7,2)) = 1$ by Lemma 3.1. Let $\{x_0\}$ be a minimum distance 2-dominating set of $G_k(7,2)$. Since $|O_i(x)| = d = 2$ for each $x \in V(G_k(7,2))$, we have $O_i(x_0) \cap O_j(x_0) = \emptyset$ for all $0 \leq i \neq j \leq 2$. On the other hand, it can be verified that for each $x \in V(G_K(7,2))$, there exist integers i, j, $0 \leq i \neq j \leq 2$, such that $O_i(x) \cap O_j(x) \neq \emptyset$ by the simply enumeration. Thus each vertex x of $G_K(7,2)$ can not form a distance 2-dominating set of $G_K(7,2)$, as claimed. By Theorem 3.1, $D = \{0,1\}$ must be a minimum distance 2-dominating set of $G_K(7,2)$.

The following result on the domination number of $G_K(n,d)$, due to Kikuchi and Shibata [16], is an immediate consequence of Lemma 3.1 and Theorem 3.1.

Corollary 3.1. ([16]) *For every generalized Kautz digraph $G_K(n,d)$, $\gamma(G_K(n,d)) = \lceil \frac{n}{d+1} \rceil$.*

It seems to be difficult to determine the minimum distance k-dominating set for general generalized Kautz digraphs $G_K(n,d)$. Now we present a sufficient condition for the distance k-domination number of $G_K(n,d)$ to be the lower bound $\lceil n/\sum_{j=0}^{k} d^j \rceil$ in Theorem 3.1.

Theorem 3.2. *For every generalized Kautz digraph $G_K(n,d)$, if $(d^{k-1} + d^k)\lceil n/\sum_{j=0}^{k} d^j \rceil \geq n$ or $d^{k-1}\lceil n/\sum_{j=0}^{k} d^j \rceil \geq \lceil \frac{n}{d+1} \rceil$ then $\gamma_k(G_K(n,d)) = \lceil n/\sum_{j=0}^{k} d^j \rceil$.*

Proof. The proof is by directly constructing a (consecutive) distance k-dominating set of $G_K(n,d)$ with cardinality $\lceil n/\sum_{j=0}^{k} d^j \rceil$. Let $D = \big\{0, 1, \cdots, \lceil n/\sum_{j=0}^{k} d^j \rceil - 1\big\}$. We claim that D is a distance k-dominating set of $G_K(n,d)$. As we have observed, if k is odd, then

$$O_{k-1}(D) = \Big\{0, 1, \cdots, d^{k-1}\Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil - 1\Big\},$$

$$O_k(D) = \Big\{n-1, n-2, \cdots, n - d^k\Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil\Big\};$$

if k is even, then

$$O_{k-1}(D) = \Big\{n-1, n-2, \cdots, n - d^{k-1}\Big\lceil n/\sum_{j=0}^{k} d^j \Big\rceil\Big\},$$

$$O_k(D) = \Big\{0, 1, \cdots, d^k\Big\lceil n\Big/\sum_{j=0}^{k} d^j \Big\rceil - 1\Big\}.$$

Clearly, $|O_{k-1}(D)| = d^{k-1}\lceil n/\sum_{j=0}^{k} d^j \rceil$ and $|O_k(D)| = d^k\lceil n/\sum_{j=0}^{k} d^j \rceil$.

Suppose that $(d^{k-1}+d^k)\lceil n/\sum_{j=0}^{k} d^j\rceil \geq n$. Note that the vertices of $O_{k-1}(D)$ and $O_k(D)$ are consecutive, so $O_{k-1}(D) \cup O_k(D) = V(G_K(n,d))$. Thus $D = \{0,1,\cdots,\lceil n/\sum_{j=0}^{k} d^j\rceil - 1\}$ is a distance k-dominating set of $G_K(n,d)$.

Suppose that $d^{k-1}\lceil n/\sum_{j=0}^{k} d^j\rceil \geq \lceil \frac{n}{d+1}\rceil$. By Lemma 3.1 and Theorem 3.1, $D_1 = \{0,1,\cdots,\lceil \frac{n}{d+1}\rceil - 1\}$ is a minimum dominating set of $G_K(n,d)$. Let $D_1' = \{n-1, n-2, \cdots, n-\lceil \frac{n}{d+1}\rceil\}$. By the definition of $G_K(n,d)$, we have $O(D_1') = \{0,1,\cdots,d\lceil \frac{n}{d+1}\rceil - 1\}$. Because $|D_1' \cup O(D_1')| = (d+1)\lceil \frac{n}{d+1}\rceil \geq n$, then D_1' is also a minimum dominating set of $G_K(n,d)$. Since the vertices of D are consecutive and $d^{k-1}\lceil n/\sum_{j=0}^{k} d^j\rceil \geq \lceil \frac{n}{d+1}\rceil$, we have either $O_{k-1}(D) \supseteq D_1$ or $O_{k-1}(D) \supseteq D_1'$. Hence $D = \{0,1,\cdots,\lceil n/\sum_{j=0}^{k} d^j\rceil - 1\}$ is a distance k-dominating set of $G_K(n,d)$. $\square$

4 Closing remarks

In this paper, we prove that the distance k-domination number of $G_B(n,d)$ takes on exactly one of two values $\lceil n/\sum_{j=0}^{k} d^j\rceil$ and $\lceil n/\sum_{j=0}^{k} d^j\rceil + 1$. In Theorems 2.2-2.4, we provide various sufficient conditions for $\gamma_k(G_B(n,d))$ equal to $\lceil n/\sum_{j=0}^{k} d^j\rceil$. It is of interest to determine the necessary and sufficient condition for $\gamma_k(G_B(n,d))$ equal to $\lceil n/\sum_{j=0}^{k} d^j\rceil$. In Theorem 3.1, we establish the sharp upper bound on $\gamma_k(G_B(n,d))$. Furthermore, we provide a sufficient conditions for $\gamma_k(G_K(n,d))$ equal to $\lceil n/\sum_{j=0}^{k} d^j\rceil$ in Theorem 3.2. We propose the following open problems.

Problem 4.1. *The sufficient condition in Theorem 2.3 is also necessary for* $\gamma_k(G_B(n,d))$ *equal to* $\lceil n/\sum_{j=0}^{k} d^j\rceil$.

For Problem 4.1, Dong, Shan and Kang [7] proved that the assertion is true for the case when $k = 1$.

Problem 4.2. *If* $G_K(n,d)$ *does not satisfy the conditions in Theorem 3.2, then* $\gamma_k(G_K(n,d)) = \lceil n \setminus (d^{k-1}+d^k)\rceil$.

For Problem 4.2, if $k = 1$, Corollary 3.1, due to Kikuchi and Shibata [16], implies that the assertion is true.

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